significant to note that the 20% increase in friction at a velocity of 213 m/sec (a realistic velocity for turbo-machinery components utilizing such liners) represents a loss in performance. Furthermore, under certain conditions there appears to be an increase in friction due to imposed noise. This conclusion is based on changes in the mean boundary layer with imposed noise; however this effect was noted only at low velocity levels ($U_\infty = 18$ m/sec).

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Similarity Solution of the Boundary-Layer Equations for Laser Heated Flows

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Introduction

RECENT developments in high-power CO_2 lasers have made it feasible to consider adding large amounts of energy to a gas flow via the absorption of 10.6μ laser radiation by inverse Bremsstrahlung and the conversion of the absorbed energy into KE. This concept can be applied to laser propulsion[1-3] and the hypersonic wind tunnel.[4] More recently the one-dimensional steady nozzle flow with laser energy addition has been examined.[5-6] When the flow medium in a laser-heated rocket thruster is pure hydrogen, which has been chosen on specific impulse considerations, the average temperature of the hot plasma core is about 14,000 K[6], and the core Reynolds number based on the nozzle throat diameter is approximately 2000. In the absence of mass injection to protect the wall, the convective heating will be severe under such high freestream temperature. To estimate the heat-transfer rate to the wall, we solve the laminar boundary-layer equations with local similarity approximation.

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Laminar boundary layers exhibiting similarity have long played an important role in exposing the principal physical features of boundary-layer phenomena and in providing a basis for approximate methods of calculating more complex, nonsimilar cases. Here, the transport properties vary significantly across the boundary layer under such extreme temperature variation (the $\rho\mu$, density-viscosity, product at the wall can be 10 times the freestream value). The existing similarity solutions with variable transport properties, i.e., Ref. 7, only cover the ratio of $\rho\mu$ product up to a factor of about 3. In addition hydrogen at 3 atm is completely dissociated at about 5000 K. Hence the boundary-layer equations with variable transport properties have been solved by a quasilinearization technique[8] and the equilibrium properties of hydrogen[9-10] have been used in the calculations.

Method of Analysis

For simplicity, we assume that Prandtl number $Pr=1$ and $\rho \sim h^{-1}$ where h is the enthalpy. A simple extension of the Cohen-Reshotko[11] analysis by including variable $\rho\mu$ will provide the following similar boundary-layer equations

$$[(\rho\mu/\rho_e\mu_e)f_{\eta\eta}]_\eta + ff_{\eta\eta} + \beta(g - f_\eta^2) = 0 \tag{1}$$

$$[(\rho\mu/\rho_e\mu_e)g_\eta] + fg_\eta = 0 \tag{2}$$

with boundary conditions $f(0) = f_\eta(0) = 0$, $g(0) = g_w$, $f_\eta(\infty) = g(\infty) = 1$ where $f_\eta = u/u_e$ is the nondimensional velocity; $g = h_s/h_{se} = (h + u^2/2)/h_{se}$ is the stagnation enthalpy ratio; and the subscripts e and s indicate the external and stagnation conditions, respectively. The similarity variables are

$$\xi = \int_0^x r^{2j}\rho_e\mu_e u_e \mathrm{d}x$$

$$\eta = (\rho_e u_e/\sqrt{2\xi})\int_0^y r^j \frac{\rho}{\rho_e}\mathrm{d}y$$

and the pressure gradient parameter is

$$\beta = \frac{2\xi}{u_e}\frac{\partial u_e}{\partial \xi}\left(\frac{h_{se}}{h_e}\right)$$

The superscripts $j=0$ and 1 correspond to the two-dimensional and the axisymmetric cases, respectively. Equations (1) and (2) can now be solved by quasilinearization

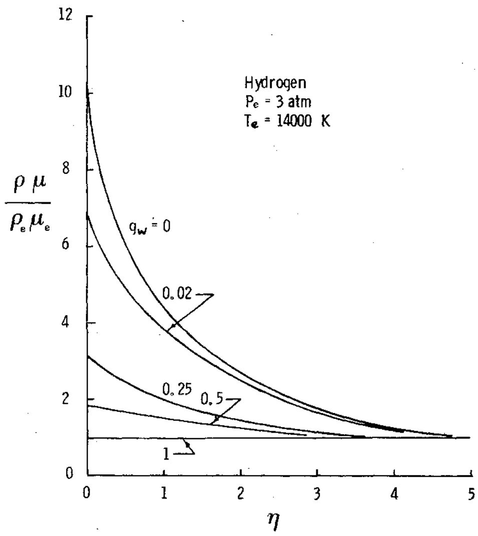

Fig. 1 Variation of density-viscosity product across the boundary layer.

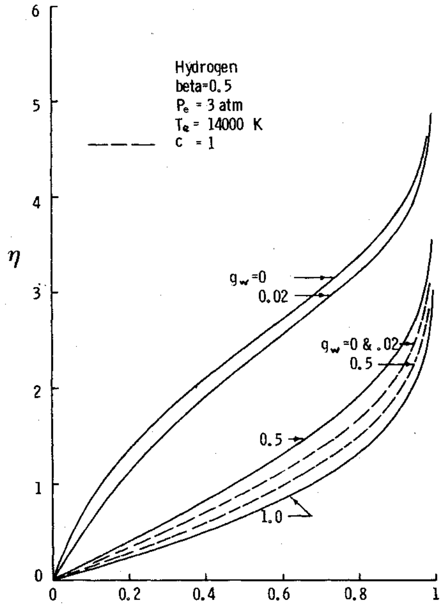

Fig. 2 Velocity profiles for $\beta = 0.5$.

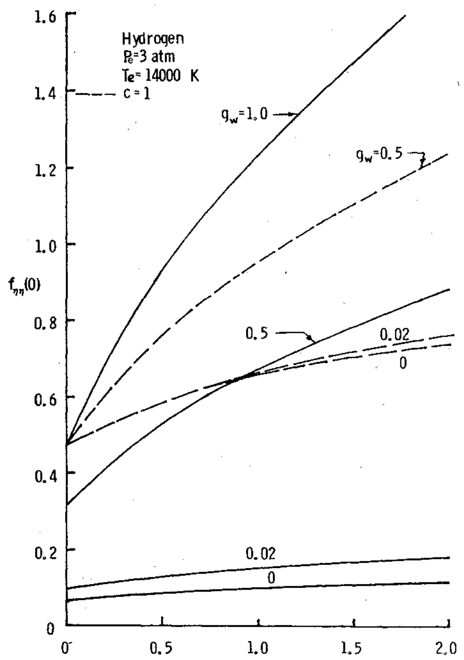

Fig. 3 The wall shear, $f_{\eta\eta}(0)$, vs the pressure gradient parameter, β.

and the resulting first-order equations are

$$(f_\eta)^k = F^k \tag{3}$$

$$(F_\eta)^k = 1/c \cdot H^k \tag{4}$$

$$(H_\eta)^k = -H/c \cdot f^k + 2\beta F \cdot F^k - f/c \cdot H^k - \beta \cdot g^k + fH/c - \beta F^2 \tag{5}$$

$$(g_\eta)^k = G^k \tag{6}$$

$$(G_\eta)^k = -G/c \cdot f^k - f/c \cdot G^k + fG/c \tag{7}$$

where $F = f_\eta$; $H = cf_{\eta\eta}$; $G = cg_\eta$; and $c = \rho\mu/\rho_e\mu_e$. Here it is assumed that functions without index are known and that functions with index k are to be determined. With the

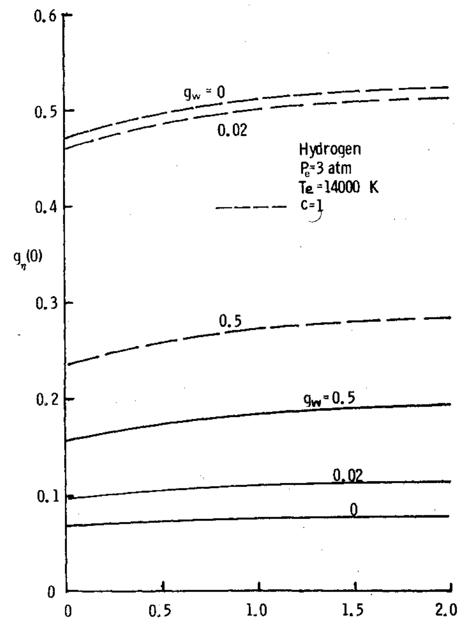

Fig. 4 The wall stagnation enthalpy gradient vs the pressure gradient parameter, β.

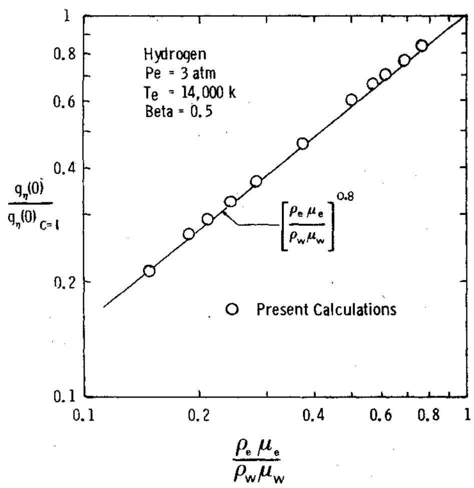

Fig. 5 The correlation of $g_\eta(0)$, as a function of the $\rho\mu$ ratio across the boundary layer.

assumed $f(\eta)$ and $g(\eta)$, one can immediately calculate the u and h profiles. The enthalpy profile, then, enables one to obtain the corresponding temperature and density distributions from Patch,[9] and consequently the viscosity profile from Yos.[10] In so doing, we have assumed that the flow is in equilibrium. Finally, Eqs. (3-7), with the boundary conditions $f^k(0) = F^k(0) = 0$, $g^k(0) = g^w$, $F^k(\infty) = g^k(\infty) = 1$, can be solved in the same manner as in Ref. 8.

Results and Discussion

Consider the external flow consisting of pure hydrogen at a temperature of 14000 K and a pressure of 3 atm. The solutions can be characterized by two parameters: the wall temperature, which provides the wall stagnation enthalpy ratio g_w, and the pressure gradient parameter β.

As shown in Fig. 1, the variation of $\rho\mu/\rho_e\mu_e$ is plotted against the normalized distance from the wall η which is zero at the wall and about 5 at the boundary-layer edge, for various values of g_w. For reference, when we take the wall temperature to be at the melting point of copper, 1300 K, g_w takes on a value of 0.01. It can be seen that c varies by nearly a factor of 7 even for $T_w = 1300$ K and increases with decreasing wall temperature. The $g_w = 0$ case corresponds to $T_w = 294$ K. The velocity profiles for $\beta = 0.5$ are shown in Fig. 2. For comparison, the constant property, $c \equiv 1$, solutions are also presented on the figure as dashed lines.

The shear τ_w and the heat flux q_w at the wall can be determined from the solution by the following expressions

$$\tau_w = (\rho_w \mu_w u_e^2 / \sqrt{2\xi}) f_{\eta\eta}(0)$$

$$q_w = (K_w \rho_w u_e h_{se} / c_{pw} \sqrt{2\xi}) g_\eta(0)$$

The results for the wall shear parameter $f_{\eta\eta}(0)$ and the wall stagnation enthalpy gradient $g_\eta(0)$, which determines the heat flux, are shown in Figs. 3 and 4, respectively, as functions of the pressure gradient parameter β. It can be seen that both quantities vary considerably from their values at constant $\rho\mu (c \equiv 1)$. As is seen in Fig. 4, $g_\eta(0)$ varies by a factor of approximately 5 for $g_w = 0.02$ case. Hence, the assumption of constant $\rho\mu$ in a hot hydrogen boundary layer would cause a significant error in calculating the heat-transfer rate. For engineering applications, the correlation of the parameter $g_\eta(0)$ as a function of the $\rho\mu$ ratio across the boundary-layer can be obtained from Fig. 4. For example, when we consider the stagnation point flow, $\beta = 0.5$, the correlation can be written as

$$g_\eta(0) = g_\eta(0) \mid_{c=1} (\rho_e \mu_e / \rho_w \mu_w)^{0.8}$$

where $g_\eta(0) \mid_{c=1}$ is the corresponding constant-property value from Ref. 11 (Fig. 5). In obtaining the above correlation, the calculations have been done for the range of g_w from 0 to 1.

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Technical Comments

Comment on "Practical Aspect of the Generalized Inverse of a Matrix"

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HASSIG[1] uses additional constraint equations to obtain a unique solution for a system of equations with more unknowns (n) than equations (m). His solution of

$$[A]_{m,n}\{x\}_{n,1} = \{y\}_{m,1}, \qquad m<n, \tag{1}$$

with the constraints

$$\{x\}_{n,1} = [A^T]_{n,m}\{\alpha\}_{m,1}, \qquad m<n, \tag{2}$$

is given as

$$\{x\}_{n,1} = [A^T]_{n,m}[[A]_{m,n}[A^T]_{n,m}]^{-1}\{y\}_{m,1}, \qquad m<n, \tag{3}$$

where $[A]_{m,n}$ has rank m.

Although Hassig's Note[1] is concerned with the generalized inverse form in Eq. (3), there are questions as to the physical meaning of the solution (3) and as to a best solution in certain applications. Since any matrix $[B]_{n,m}$ with rank m could be used in place of $[A^T]_{n,m}$ in Eqs. (2) and (3), what is special about $[A^T]_{n,m}$?

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If Eq. (1) is substituted into in Eq. (3) for $\{y\}$, then

$$[K]_{n,n}\{x\}_{n,1} = \{0\}_{n,1} \tag{4}$$

$$[K]_{n,n} = [[I] - [A^T][[A][A^T]]^{-1}[A]]_{n,n} \tag{5}$$

Since $[K]_{n,n}$ is not zero but has rank $n-m$, Eq. (4) imposes $n-m$ explicit independent constraint equations directly on the x_i. These equations could be combined directly with Eq. (1) to give the $\{x\}$ solution. It is evident that these same $n-m$ equations are also the conditions to make the sum of the squares of the x_i a minimum, or

$$\sum_{i=1}^{n} (x_i)^2 = \text{minimum} \tag{6}$$

a result noted by Greville.[2] Thus, the physical meaning of using $[A^T]_{n,m}$ in Eqs. (2) and (3) is to obtain the set of x_i satisfying Eq. (6). No other $[B]_{n,m}$ matrix in place of $[A^T]_{n,m}$ will do this. However, is this solution (3), restrained by condition (6), a best solution in actual applications?

Consider the example of representing a function over an interval with a set of n known functions with unknown coefficients x_i. If m collocation points ($m<n$) are used to determine the coefficients, Eq. (1) results. The best values for the x_i are those that give the best representation of the function in the interval. In the case of using polynomials to represent the exponential function, the solution (3) is fair but it is easy to select $[B]$ matrices in place of $[A^T]$ that give better results for the x_i.

In redundant structural truss problems, Eq. (1) gives the equilibrium equations and Eq. (2), with a $[B]$ matrix involving geometry and material properities, gives the deflec-